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The Hausdorff Metric and Čebyšev Centres

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1. INTRODUCTION

In this paper we examine the relationship between the Hausdorff distance between convex sets in a normed linear space and the distance between their Čebyšev centres. We show that a spectrum of different behaviours is possible depending upon the orientation of the sets and the nature of the norm on the space. Recently a very special case of this problem has been discussed in the problem section of the *American Mathematical Monthly* [1].

2. PRELIMINARIES

Let A, B be convex, bounded nonempty sets in a normed linear space X. We define the Hausdorff distance between A and B [3], by

$$D(A, B) = \max\{\sup\{d(x, A): x \in B\}, \sup\{d(y, B): y \in A\}\},$$

where as usual for $S \subset X$, $d(x, S) = \inf\{\|x - s\|: s \in S\}$. We define a Čebyšev centre of a bounded set $S \subset X$ with respect to $T \subset X$, written $c(S, T)$, as the centre of a closed ball of minimum radius that contains S with centre in T. This ball is called the Čebyšev ball. If $T = S$ we sometimes call the Čebyšev centre of S in T simply the centre of S, written c_S. The corresponding radii will be denoted by $\gamma(S, T)$ and $\gamma(S, S) = \gamma_s$. Define $R(A, B)$ by

$$R(A, B) = \frac{\|C_A - C_B\|}{D(A, B)}.$$

for $A \neq B$. Now define $\mu(\mathscr{F}) = \sup\{R(A, B): (A, B) \in \mathscr{F}\}$, where $\mathscr{F}$ is an as yet unspecified family of pairs of bounded, convex, nonempty subsets of X.

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We assume that all the subsets appearing in $\mathscr{F}$ have unique centres. This will always be the case in a strictly convex finite dimensional space. See also [5] for infinite dimensional criteria. The value

$$\lambda(\mathscr{F}) = \inf\{R(A, B): (A, B) \in \mathscr{F}\}$$

is in general not as useful in relating the Hausdorff and Čebyšev properties of the space. The families $\mathscr{F}$ that we will consider are

$$\mathscr{F}_b = \{(A, B): A, B \text{ are closed, bounded, nonempty, convex subsets of } X\},$$

$$\mathscr{F}_1 = \{(A, B): (A, B) \in \mathscr{F}_b,\ N(c_A, r_A) \cap N(c_B, r_B) = \varnothing\},$$

(where $N(x, r) \equiv \{y \in X: \| y - x \| < r\}$),

$$\mathscr{F}_2 = \{(A, B): (A, B) \in \mathscr{F}_b,\ c_A \notin B, c_B \notin A\},$$

and

$$\mathscr{F}_3 = \{(A, B): (A, B) \in \mathscr{F}_b,\ A \cap B = \varnothing\}.$$

Note that

$$\mathscr{F}_b \supset \mathscr{F}_i, \qquad i = 1, 2, 3 \tag{1}$$

and

$$\mathscr{F}_1 \subset \mathscr{F}_i, \qquad i = 2, 3. \tag{2}$$

In the following sections, we will tacitly assume that all spaces considered are at least of dimension 2. In many arguments two dimensional subspaces of X will be considered and a superscript 2 on a set will indicate this. For instance $\bar{N}^2(x, r)$ will be $\bar{N}(x, r)$ intersected with the appropriate two dimensional subspace.

3. Some Properties of Čebyšev Centres

We begin with the following theorem. The equivalence of (b) and (d) in the theorem was shown by Garkavi in [6].

THEOREM 1. *Suppose X is a Banach space in which closed convex and bounded sets have unique Čebyšev centres (in themselves). The following are equivalent*:

(a) *For every pair of closed bounded convex sets A and B with $A \subset B$ one has $r(A, A) \leqslant r(B, B)$.*

(b) *For every closed bounded convex set A in X, $c(A, X) \in A$.*

(c) *Every two dimensional cross section S of the unit ball has radius $r(S, S) = 1$.*

(d) *X is two dimensional or is an inner product space.*

Proof. If (b) fails for A, the ball $B = B(c(A, X), r(A, X))$ contains A and $r(A, A) > r(A, X) = r(B, B)$. Thus (a) fails and so (a) impiles (b). Similarly, if (c) fails for some cross section S, $r(S, S) > 1 \geq r(S, X)$, and (b) fails. Thus (b) implies (c).

Suppose that (c) holds and that X has dimension at least three. Let $X_0 \subset X$ have dimension three. Since X is an inner product space if every three dimensional subspace is, and since every two dimensional cross section of X_0 is a two dimensional cross section of X, one may assume that the dimension of X is three. Let Π be a plane in X_0 defined by $\Pi = \{x: f(x) = 0\}$ for some linear function f. We will show that X has a norm one projection onto Π and it will follow from Kakutani's theorem [9, 10] that X is Euclidean. To this end, let B be the unit ball in X and let $B_0 = B \cap \Pi$ and $B_n = B \cap f^{-1}(1/n)$. Let c_n be the centre of B_n. If $b_0 \in B_0$ with $\|b_0\| = 1$ then $\|c_n - (c_n - b_0)\| = 1$.

Since $c_n - b_0 \in B_n$ and $r(B_n) \leq 1$, $\|c_n - b_0\| \geq 1$. It follows that, for $t \geq 1$, $\|tc_n - b_0\| \geq 1$ because

$$\frac{1}{t}\|tc_n - b_0\| + \left(1 - \frac{1}{t}\right)\|b_0\| \geq \|c_n - b_0\| \geq 1.$$

Let $c_n/\|c_n\| = k_n$. Then k_n can be supposed convergent to k_0 with $\|k_0\| = 1$. Thus for all b_0 in B_0 with $\|b_0\| = 1$ one has

$$\|tk_n + b_0\| \geq 1 \qquad \text{if} \quad t \geq \|c_n\|.$$

We will show below that $\|c_n\| \to 0$. Assuming this it follows, on letting $n \to \infty$ and invoking the symmetry of B_0, that for all $t \in R$ and $\pi \in \Pi$

$$\|tk_0 + \pi\| \geq \|\pi\|.$$

This clearly excludes k_0 lying in Π and we may define a projection P of X onto Π by setting

$$Px = \pi$$

if x is written (uniquely) as $tk_0 + \pi$, $\pi \in \Pi$. This is clearly a norm one projection.

Finally we show that $c_n \to 0$. To this end we may suppose $c_n \to c_0 \in B_0$.

Let $b \in B_0$ and let $b_n = ((n-1)/n)b + (1/n)c_1$. Then $b_n \in B_n$ and

$$1 \geqslant \| c_n - b_n \| = \left\| \frac{n-1}{n} b + \frac{1}{n} c_1 - c_n \right\|.$$

On taking limits one sees that $\| b - c_0 \| \leqslant 1$ for all b in B_0. Since B_0 is strictly convex this implies that $c_0 = 0$ and (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (a). Suppose $A \subset B$ and $r(A, A) > r(B, B)$. On replacing B by $B(c(B, B), r(B, B))$ and normalizing, we may assume that $A \subset B(0, 1)$, $r(A, X) = 1$ and $r(A, A) > 1$.

Let $\bar{a}$ be the (unique) nearest point to zero in A. Then, in a Hilbert space, it follows that $(\bar{a} - a) \cdot \bar{a} \geqslant 0$ for each a in A so that

$$\| \bar{a} - a \|^2 \leqslant \| \bar{a} - a \| \, \| a \|$$

and $\| \bar{a} - a \| \leqslant \| a - 0 \|$, which contradicts $r(A, A) > 1$.

Finally, if X is two dimensional one verifies that the midpoint m of the chord of $B(0, 1)$ tangent to A at $\bar{a}$ satisfies

$$\| m - a \| \leqslant 1 \qquad \forall a \in A.$$

Since X is two dimensional and strictly convex, it is uniformly convex and so $\sup_{a \in A} \| m/2 - a \| < 1$. Thus $r(A, X) < 1$ which again is a contradiction. Thus (d) $\Rightarrow$ (a). ∎

We note that only for (d) $\Rightarrow$ (a) did we use the completeness of X.

As the above theorem indicates, in any non-Euclidean three dimensional space, many desirable properties of Čebyšev centres do not hold. We illustrate this by explicitly constructing a two dimensional cross section of the l_p unit ball in $\mathbb{R}^3$ ($p \neq 2$), with radius greater than one. The following proposition will be useful in calculating this radius.

PROPOSITION 1. *Let X be a strictly convex normed linear space and T an isometry such that for some point $\bar{x}$, $T^n \bar{x} = \bar{x}$. Then the centre of the simplex $S = co(\bar{x}, T(\bar{x}), \ldots, T^{n-1}(\bar{x}))$ is its barycentre.*

Proof. Let $x_0 = \sum_{i=0}^{n-1} t_i T^i(\bar{x})$ with $t_i \geqslant 0$, $i = 1, 2, \ldots, n$ and $\sum_{i=0}^{n-1} t_i = 1$. Let g be defined by

$$g(x) = \frac{1}{n} \sum_{i=0}^{n-1} \| x - T^i(\bar{x}) \|^2 \qquad \forall x \in X.$$

Then g is a strictly convex function since the norm is strictly convex. Now

$$\min_{x \in S} \max_{s \in S} \| x - s \|^2 = \min_{x \in S} \max_{0 \leqslant i \leqslant n} \| x - T^i(\bar{x}) \|^2$$
$$\geqslant \min_{x \in S} g(x).$$

Suppose that x_0 minimizes g over S. Then for $1 \leqslant i \leqslant n-1$, $g(x_0) = g(T^i(x_0))$ and if $\hat{x} = (1/n)\sum_{i=0}^{n-1} T^i(x_0)$ one has

$$g(\hat{x}) \leqslant \frac{1}{n}\sum_{i=0}^{n-1} g(T^i(x_0)) = g(x_0) \leqslant g(\hat{x}).$$

This implies that $x_0 = \hat{x}$ and, in turn, that

$$x_0 = \sum_{i=0}^{n-1} \frac{1}{n} T^i(x_0) = \sum_{i=0}^{n-1} \frac{1}{n} T^i\left(\sum_{j=0}^{n-1} t_j T^j(\bar{x})\right)$$
$$= \frac{1}{n}\sum_{j=0}^{n-1} T^j(\bar{x}).$$

Thus the barycentre minimizes g over S. Then

$$\min_{x\in S}\max_{s\in S} \| x - s \|^2 \geqslant g\left(\sum_{i=0}^{n-1} \frac{1}{n} T^i(\bar{x})\right)$$
$$= \frac{1}{n}\sum_{i=0}^{n-1} \| x_0 - T^i(\bar{x}) \|^2$$
$$= \max_{0\leqslant i\leqslant n-1} \| x_0 - T^i(\bar{x}) \|^2 = \max_{s\in S} \| x_0 - s \|^2,$$

because $\| x_0 - \bar{x} \| = \| T^i x_0 - T^i \bar{x} \| = \| x_0 - T^i \bar{x} \|$ for each i, as T is an isometry. This shows that x_0 is the Čebyšev centre for S in S. ∎

Now in $l_p(3)$ consider $T(x_1, x_2, x_3) = (x_2, x_3, x_1)$. Then T is an isometry and we will use the proposition to show explicitly that cross sections of the l_p unit ball ($p \neq 2$) can be constructed of radius larger than one. This will provide examples to parts (b), (c), (d) of Theorem 1. We consider two cases.

Case one. $1 < p < 2$. Let $x(d) = (2d + 3, -d, -d)$ for $d > 0$. Then $T^3 x(d) = x(d)$ and the simplex $S(d) = co\{x(d), T(x(d)), T^2(x(d))\}$ is a triangle in the plane $x_1 + x_2 + x_3 = 3$. We show that for d sufficiently large $S(d)$ has radius greater than $\| x(d) \|_p$. Then it follows easily (directly or from Theorem 1) that the plane cross section

$$\{(x_1, x_2, x_3) \colon \|(x_1, x_2, x_3)\|_p \leqslant 1, \quad x_1 + x_2 + x_3 = 3 \| x(d) \|_p^{-1}\}$$

is the requisite cross section.

By the above proposition the centre of $S(d)$ is the barycentre $(1, 1, 1)$. Thus $S(d)$ has radius $((2d + 2)^p + 2(d + 1)^p)^{1/p}$ and it suffices to show that, for d sufficiently large,

$$(2d + 3)^p - (2d + 2)^p < 2[(d + 1)^p - d^p].$$

On applying the Mean Value Theorem to each side of this inequality we have

$$(2d + 3)^p - (2d + 2)^p \leqslant p(2d + 3)^{p-1}; \qquad 2(d + 1)^p - d^p \geqslant 2d^{p-1},$$

and, since for $1 < p < 2$ one has $(2d + 3)^{p-1} < 2d^{p-1}$ for d sufficiently large, the inequality is established.

Case two. $2 < p < \infty$. We now let $x(d) = (3 - 2d, d, d)$ and consider $d > 3/2$. We now wish to establish that $|3 - 2d|^p + 2d^p < |2 - 2d|^p + 2|d - 1|^p$ or

$$(2d - 2)^p - (2d - 3)^p > 2d^p - 2(d - 1)^p.$$

Again applications of the Mean Value Theorem reduce this to showing that $(2d - 3)^{p-1} > 2d^{p-1}$. For d large enough this follows since $p > 2$.

Thus for each $p \neq 2$ one finds that a suitable cross section lies in the plane $x_1 + x_2 + x_3 = c(p)$ for some positive constant $c(p)$.

In case $p = 1$, one may simply take the face generated by $x(0) = (3, 0, 0)$. In case $p = \infty$, $x(3) = (-3, 3, 3)$ generates an appropriate cross section.

We complete this section with a characterization of Čebyšev balls in Hilbert space.

PROPOSITION 2. *Let $\mathscr{H}$ be a Hilbert space and K a closed, bounded, convex, nonempty subset of $\mathscr{H}$. Then $\overline{N}(x, r) \supset K$ is the Čebyšev ball for K if and only if $x \in \hat{K} = \overline{co}\{K \cap N^*(x, r)\}$ (where $N^*(x, r) = N^*$ is the boundary of $\overline{N}(x, r) = \overline{N}$).*

Proof. Assume without loss that $x = 0$, $r > 0$ and that $\overline{N}$ is the Čebyšev ball for K. Suppose $0 \notin \hat{K}$. Let H be a hyperplane separating $\hat{K}$ and 0 with $d(0, H) > 0$ and $H \cap \hat{K} = \varnothing$. Denote the open half-space determined by H and containing $\hat{K}$ by H^- with $\sim H^- = H^+$. Let p have smallest norm in H. Let $S^- = H^- \cap K$ and $\delta = d(S^-, N^*) > 0$. Let $p' \in (0, p]$ with $\|p'\| = \delta/2$. Choose y arbitrarily in $S^+ = H^+ \cap K$. Let P be a two dimensional subspace of $\mathscr{H}$ containing y and p'. Represent points in P using a rectangular coordinate system with basis chosen so that $\overline{N}^2$ is a Euclidean disk. The line H^2 is orthogonal to the line through 0 and p'. A simple geometric argument in P shows that $y \in \overline{N}(p', (1 - \delta^2/4)^{1/2})$ so $\overline{N}(p', (1 - \delta^2/4)^{1/2}) \supset S^+$. By the triangle inequality, $\overline{N}(p', r - \delta/2) \supset S^-$. Let $\rho = \max\{r - \delta/2, (1 - \delta^2/4)^{1/2}\} < r$. Then $\overline{N}(p', \rho) \supset K$, contradicting the assumption that $\overline{N}$ is a Čebyšev ball.

Conversely, assume $0 \in \hat{K}$. It suffices to show that if K is perturbed by any $v \neq 0$, that $(\hat{K} - v) \cap \sim\overline{N} \neq \varnothing$. Let H be the hyperplane containing 0, orthogonal to v and let H^+ be the closed half-space determined by H and containing v. Since $0 \in \hat{K}$ we may deduce the existence of $y \in H^+ \cap N^* \cap \hat{K}$.

But by looking at a cross section, it is clear that $y - v \in \overline{N}$ so $(K - v) \cap \sim\overline{N} = \emptyset$. ∎

4. Calculation of Bounds

We now calculate bounds for $\mu(\mathscr{F}_b)$, $\mu(\mathscr{F}_1)$, $\mu(\mathscr{F}_2)$ and a lower bound for $\mu(\mathscr{F}_3)$.

THEOREM 2. *For any space X with subsets appearing in $\mathscr{F}$ having unique Čebyšev centres, we have $\mu(\mathscr{F}_b) = \mu(\mathscr{F}_2) = \infty$.*

Proof. Let $S \subset X$ be a two dimensional subspace of X, representing points in S using a rectangular coordinate system UV. Let $\overline{N}^2 = S \cap \overline{N}(0, 1)$. Let D be the maximal, closed Euclidean disc, centre 0, contained in $\overline{N}^2$. Let $z = (z_u, z_v) \in D \cap N^*$ with N^* the boundary of $\overline{N}^2$. Assume without loss that $z_u = 0$. Note that $-z \in D \cap N^*$ and z and $-z$ are points of smoothness of $\overline{N}^2$. Also note that the supporting tangent lines at z and $-z$ are parallel to the U axis. Let $\|\cdot\|_E$ denote the norm induced on S by D. Given $0 < \epsilon < 1$, let $w = w(\epsilon) = (w_u, 0)$ be such that $\|w\|_E = \epsilon$, $w_u < 0$. Define $K' = K'(\epsilon)$ and $K'' = K''(\epsilon)$ by

$$K' = \{s = (s_u, s_v) \in \overline{N}^2 : w_u \leq s_u \leq 0\}$$

and

$$K'' = \{s = (s_u, s_v) \in \overline{N}^2 : 0 \leq s_u \leq -w_u\} + w.$$

Clearly 0 is the Čebyšev centre of K' and w the Čebyšev centre of K'', the distance between them being $\|w\|$. Let R be the closed rectangle in S with vertices at z, $-z$, $z - w$ and $-z - w$. Let $Q = R \sim D$. Notice that $Q \supset K'' \sim K'$. Then

$$\begin{aligned}
&\sup_{x \in K'' \sim K'} \left[\inf_{s \in K'} \|x - s\|\right] \\
&\quad \leq \sup_{x \in Q} \left[\inf_{s \in K'} \|x - s\|\right] \\
&\quad \leq \sup_{x \in Q} \left[\inf_{s \in K'} \alpha_1 \|x - s\|_E\right] \\
&\quad = \alpha_1 \sup_{x \in Q} \left[\inf_{s \in K'} \|x - s\|_E\right] \leq \alpha_1 \sup_{x \in Q} \left[\inf_{s \in R \cap D} \|x - s\|_E\right] \\
&\quad \leq \alpha_1 [1 - (1 - \epsilon^2)^{1/2}],
\end{aligned}$$

where $\alpha_1 > 0$ is independent of ϵ.

Interchanging the roles of K' and K'' we obtain

$$\sup_{x \in K'-K''} [\inf_{s \in K''} \| x - s \|] \leqslant \alpha_1[1 - (1 - \epsilon^2)^{1/2}],$$

so $D(K', K'') \leqslant \alpha_1[1 - (1 - \epsilon^2)^{1/2}]$. Also, there is an $\alpha_2 > 0$ independent of ϵ such that $\| w \| \geqslant \alpha_2 \| w \|_E = \alpha_2 \epsilon$, so

$$\frac{\| w \|}{D(K', K'')} \geqslant \frac{\alpha_2 \epsilon}{\alpha_1[1 - (1 - \epsilon^2)^{1/2}]} \to \infty$$

as $\epsilon \to 0$. Thus $\mu(\mathscr{F}_b) = \infty$. As slight perturbation of K' and K'' will allow them to satisfy the criteria for (K', K'') to be in $\mathscr{F}_2$ and we may deduce that $\mu(\mathscr{F}_2) = \infty$ as well. ∎

THEOREM 3. *For any X as in Theorem 2, $\mu(\mathscr{F}_3) \geqslant 2$.*

Proof. We modify the construction in Theorem 2. Using the notation in that theorem, let l be the line through $w/2$, parallel to the V axis, and let $\hat{K}'$ be the maximal closed subset of K', containing 0 and supported by l. Define $\hat{K}''$ similarly. 0 is the Čebyšev centre of $\hat{K}'$ and w the Čebyšev centre of $\hat{K}''$. By comparison to R and D and using the continuity of the norm we may deduce that

$$\lim_{\epsilon \to 0} \frac{\| w \|}{D(\hat{K}', \hat{K}'')} \geqslant 2.$$

A slight perturbation of $\hat{K}'$ and $\hat{K}''$ will give sets that fail to intersect and we conclude that $\mu(\mathscr{F}_3) \geqslant 2$. ∎

Theorem 2 demonstrates that $\mu(\mathscr{F}_b)$ and $\mu(\mathscr{F}_2)$ are not sensitive measures of the norm's geometry. It is not clear how useful $\mu(\mathscr{F}_3)$ may be. However the situation is quite different for $\mu(\mathscr{F}_1)$. In this case one is asking the following question: Given two arbitrary disjoint balls, how can you choose convex sets A, B in each ball so that the balls are the Čebyšev balls for the convex sets and the Hausdorff distance between the two sets is minimized? Having done this, what placement of balls maxiimzes the ratio $R(A, B)$?

THEOREM 4. *For any strictly convex normed linear space X, satisfying the conditions of Theorem 2, $(1 + 5^{1/2})/2 \leqslant \mu(\mathscr{F}_1) \leqslant 2$.*

Proof. We perform the following construction in a two dimensional subspace of X, with points referenced using a rectangular coordinate system. We may parameterize the boundary of $\overline{N}^2(0, 2)$ by $y(t) = (y_1(t), y_2(t))$, $t \in [0, 1]$ and the curve bounding the unit ball by $x(t) = (x_1(t), x_2(t))$, $t \in [0, 1]$ in such a way that $\| x(t) + y(t) \| = \| x(t) - y(t) \|$ for each $t \in [0, 1]$.

One way to show this is to consider the locus of points equidistant from x and $-x$, and find a point on the locus of norm 2 (exactly two such points by strict convexity), then allow x to move counterclockwise along the boundary of the unit ball and observe that y changes with x, continuously and also counterclockwise. Let $A = \int_0^1 x_1\, dx_2$, the area contained by the curve $x = x(t)$. Clearly $4A = \int_0^1 y_1\, dy_2$. Suppose $z = z^-(t) = x(t) - y(t)$, $t \in [0, 1]$ is contained in the interior of the ball, centre 0, radius $5^{1/2}$. Then so is $z = z^+(t) = x(t) + y(t)$, $t \in [0, 1]$. Therefore $\int_0^1 (x_1 - y_1)\, d(x_2 - y_2) + \int_0^1 (x_1 + y_1)\, d(x_2 + y_2) < 5A + 5A = 10A$. But this sum of integrals is also equal to $2[\int_0^1 x_1\, dx_2 + \int_0^1 y_1\, dy_2] = 10A$, a contradiction. By assuming that $\|z^+(t)\| > 5^{1/2}$ for all t, a similar argument also gives a contradiction. So let x_0, y_0 be such that $\|x_0\| = 1$, $\|y_0\| = 2$, $\|x_0 - y_0\| = \|x_0 + y_0\| = 5^{1/2}$. Let A be the half-ball centre 0, determined by the line segment $[-x_0, x_0]$ and B the half-ball, centre y_0, determined by $[y_0 - x_0, y_0 + x_0]$, chosen so these half-balls touch at $\frac{1}{2}y_0$. The distance between the centres of these balls is clearly $\|y_0\| = 2$. The Hausdorff distance $D(A, B)$ is readily computed as $5^{1/2} - 1$. So

$$\frac{\|y_0\|}{D(A, B)} = \frac{2}{5^{1/2} - 1} = \frac{5^{1/2} + 1}{2}.$$

We conclude that $\mu(\mathscr{F}_1) \geqslant (1 + 5^{1/2})/2$, the golden mean, which we denote by m. The upper bound is a simple consequence of the triangle inequality. ∎

In a Hilbert space, the lower bound is actually attained.

THEOREM 5. *If X is a Hilbert space, then $\mu(\mathscr{F}_1) = m$.*

Proof. Let B_1, B_2 be two disjoint closed balls in X. Assume B_1 has centre 0, radius 1, and B_2 has centre x_2, radius r_2, $0 < r_2 \leqslant 1$. Let (K_1, K_2) be a pair of closed, bounded, convex, nonempty subsets with B_i the Čebyšev ball of K_i, $i = 1, 2$. Then $D(K_1, K_2) \geqslant (1 + \|x_2\|^2)^{1/2} - r_2$. Indeed, let H_1 be the hyperplane through 0, orthogonal to x_2 and let H_2 contain x_2 and be parallel to H_1. Let H_1^+, H_1^- be the two closed half-spaces determined by H_1, where $x_2 \in H_1^+$. Similarly define H_2^+, H_2^- with $0 \in H_2^-$. By Proposition 2, there is a $y \in B_1^* \cap H_1^- \cap K_1$, where B_1^* is the boundary of B_1. Referring to a two dimensional subspace P containing y and x_2 it is seen that $d(y, K_2) \geqslant d(y, B_2) \geqslant d(y^*, B_2) = (1 + \|x_2\|^2)^{1/2} - r_2$, where y^* is the appropriate point in $P \cap H_1 \cap B_1^*$, so $D(K_1, K_2) \geqslant (1 + \|x_2\|^2)^{1/2} - r_2$. To generalize, redefine K_1, K_2 as closed, bounded, nonempty, convex sets with disjoint Čebyšev balls $\bar{N}(t_1, r_1)$, $\bar{N}(t_2, r_2)$, $r_1 \geqslant r_2$. Let $r = r_2/r_1$ and $\delta = \|t_1 - t_2\|/r_1$. Then by the above argument, we may deduce that

$$\frac{\|t_1 - t_2\|}{D(K_1, K_2)} \leqslant \frac{\delta}{(1 + \delta^2)^{1/2} - r}.$$

Thus

$$\sup_{(K_1,K_2)\in\mathscr{F}_1} \frac{|t_1 - t_2|}{D(K_1, K_2)} \leqslant \sup \left\{ \frac{\delta}{(1+\delta^2)^{1/2} - \imath} : \imath - 1 \leqslant \delta, 0 \leqslant \imath \leqslant 1 \right\}.$$

Standard techniques give a global maximum for $\delta/((1 + \delta^2)^{1/2} - \imath)$ on the given region at $\imath = 1$, $\delta = 2$, so $\mu(\mathscr{F}_1) \leqslant m$. To complete the argument, we may refer to the previous theorem, or note that for $K_1 = \bar{N}(0, 1)$, $K_2 = \bar{N}(x_2, 1 - \epsilon)$ with $\|x_2\| = 2$, $0 < \epsilon < 1$ we may deduce that

$$\frac{\|x_2\|}{D(K_1, K_2)} = m + g(\epsilon)$$

where $g(\epsilon) < 0$ and $g(\epsilon) \to 0$ as $\epsilon \to 0$. Thus $\mu(\mathscr{F}_1) = m$. ∎

Theorem 5 shows that the lower bound given in Theorem 4 is best possible. Calculations in $l_p(2)$ as $p \to \infty$ shows that the upper bound is also best possible.

Remark 1. Does $\mu(\mathscr{F}_1) = m$ imply that X is Euclidean? The integral technique used in the proof of Theorem 4 shows that it would be sufficient to demonstrate that $\|x\| = \|y\| = 1$ and $\|x - 2y\| = \|x + 2y\| = 5^{1/2}$ implies X is Euclidean. There is some reason for believing this is possible since it is known [4, 7] that if the restriction on the norms of x and y is removed, the condition characterizes Euclidean space. In James' terminology [7], we are asking whether a localized version of his results on Pythagorean and isosceles orthogonality holds.

Remark 2. Clearly the same integration technique of Theorem 4 can be used to establish a variety of bounds on $R(A, B)$ for other choices of $\mathscr{F}$. For example, we could require c_A not to be an element of the Čebyšev ball for B and vice versa.

5. Conclusions and Open Questions

There are some additional questions that arise from the above discussion. First. is it true that $\mu(\mathscr{F}_3) = 2$ in all spaces? Second, does $\mu(\mathscr{F}_1) = m$ characterize inner product spaces? More generally, in light of Remark 1, does the following condition characterize inner product spaces? For fixed $\lambda > 0$:

$$\begin{aligned} &\|x\| = \|y\| = 1, \\ &\|x + \lambda y\| = \|x - \lambda y\| \Rightarrow \|x + \lambda y\|^2 = \|x - \lambda y\|^2 \\ &= \|x\|^2 + \lambda^2 \|y\|^2 = 1 + \lambda^2. \end{aligned} \qquad (*)$$

Our constant $\mu(\mathscr{F}_1)$ is analogous to the rectangular constant of Joly [2, 8]. Essentially, we consider isoceles orthogonality while Joly considers Birkhoff orthogonality. This analogy also suggests that it might be possible to show that (*) makes Birkhoff orthogonality symmetric and so, in spaces of dimension at least 3, characterizes Euclidean space.

Finally, we observe that all our limiting constructions provide sets in which the centre in the space lies in the set. It follows that all the results in Section 4 hold (except for $u(F_1) \leq 2$) if c_A, c_B are replaced by $c(A, X)$ and $c(B, X)$.

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References

1. Advanced Problem Solution 6063 [1975, 1016], *Amer. Math. Monthly* **84** (1977), 578–579.
2. C. Benítez and M. del Rio, The rectangular constant for two-dimensional spaces, *J. Approximation Theory* **19** (1977), 15–21.
3. G. Choquet, "Lectures on Analysis," Benjamin, New York, 1969.
4. M. M. Day, "Normed Linear Spaces," 3rd ed., Springer-Verlag, New York/Heidelberg/Berlin, 1973.
5. A. L. Garkavi, The best possible net and the best possible cross section of a set in a normed space, *Izv. Akad. Nauk SSSR Ser. Mat.* **26** (1962), 87–106.
6. A. L. Garkavi, On the Čebyšev center and convex hull of a set, *Uspehi Mat. Nauk* **19**, No. 6 (120) (1964), 139–145.
7. R. C. James, Orthogonality in normed linear spaces, *Duke Math. J.* **12** (1945), 291–302.
8. J. L. Joly, Caractérisation d'espace hilbertiens au moyen de la constante rectangle, *J. Approximation Theory* **2** (1969), 301–311.
9. S. Kakutani, Some characterizations of Euclidean space, *Japan J. Math.* **16** (1939), 93–97.
10. R. S. Phillips, A characterization of Euclidean spaces, *Bull. Amer. Math. Soc.* **46** (1940), 930–933.